Chapter-V
Compact Metric Spaces

Definition: Let (X, d) be a metric space. Then a family $\{G_\alpha : \alpha \in \Delta$, an index set$\}$ is said to be an open cover for X if $X \subseteq \bigcup_{\alpha \in \Delta} G_\alpha$, where each G_α is an open set.

Example: The family $\{(n-2, n+2) : n = 0, \pm 1, \pm 2, \ldots\}$ of open intervals and have open sets form an open cover for R become $R = \bigcup_{n \in Z} (n-2, n+2)$

Example: The family $\{(n-\frac{1}{2}, n+\frac{1}{2}) : n = 0, \pm 1, \pm 2, \ldots\}$ of open intervals and hence open sets in the real number space does not form an open cover.

Definition: A metric space (X, d) is said to be compact if every open cover of X has a finite subcover.

The real number space R with respect to the usual metric is not compact because the open cover $\{(n-2, n+2) : n = 0, \pm 1, \pm 2, \ldots\}$ for R does not admit any finite sub cover for R.

Definition: A subset E of a metric space (X, d) is said to be a compact subset of X if as a subspace E becomes a compact metric space.

Theorem: Let (X, d) be a compact metric space and G be a closed subset of X. Then G is compact subset of X.

proof: Let $\{G_\alpha : \alpha \in \Delta\}$ be an open cover for G. Since each G_α is open relative to G, $\exists$ an open set H_α in (X, d) such that $G_\alpha = H_\alpha \cap G$, $\alpha \in \Delta$. Since G is closed, $X - G$ is open. Hence the family $\{H_\alpha : \alpha \in \Delta\}$ together with $X - G$ forms an open cover for X. By compactness of (X, d), $\exists$ a finite number of members of the family $\{H_\alpha : \alpha \in \Delta\}$, say $H_1, H_2, \ldots, H_n$ and possible $X - G$ to cover X. Hence, X is a subset of $H_1 \cup H_2 \cup \ldots H_n \cup (X - G)$ i.e;

$$G \subseteq H_1 \cup H_2 \ldots \cup H_n \text{ i.e;}$$

$$G \subseteq (H_1 \cap G) \cup (H_2 \cap G) \ldots \cup (H_n \cap G) \text{ i.e;}$$

$$G \subseteq G_1 \cup G_2 \cup \ldots \cup G_n$$

So, G is compact in (X, d).

Definition: A metric space (X,d) is said to be sequentially compact if every sequence in (X,d) has a convergent subsequence in (X,d).

Definition: A metric space (X,d) is said to have Bolzano-Weierstrass property (B-W property) if every infinite subset of X has a limiting point.

Theorem: In a metric space (X,d) the following are equivalent:

(i) The space (X,d) is sequentially compact.

(ii) The space (X,d) has B-W property.

Proof: (i) $\Rightarrow$ (ii). Let (X,d) be sequentially compact and A be an infinite subset of X.

Since, A is an infinite subset of X, $\exists$ a sequence $\{x_n\}$ of distinct points in A. Since $\{x_n\}$ is a sequence in X, by sequential compactness of X, $\{x_n\}$ has a convergent subsequence say $\{x_{n_k}\}$. Let $\lim_{k\to\infty} x_{n_k} = u \in X$. Then u is a limiting point of A because $x_{n_k} \in A$, $\forall\, k = 1,2,\ldots$. Hence (X,d) has B-W property. So (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i). Let (X,d) has B-W property and $\{x_n\}$ be a sequence in (X,d). If one of the terms of $\{x_n\}$ is repeated infinitely many times. Then perusing that term we produce a subsequence of $\{x_n\}$ which is convergent. Else we take the elements of $\{x_n\}$ to be distinct. Then $A = \{x_1, x_2, \ldots, x_n, \ldots\}$ becomes an infinite subset of X. So by (ii) A has a limiting point, say $u \in X$. Let $0 < r_1 < 1$. Then $S_{r_1}(u) \cap (A - \{u\}) \neq \phi$ because u is a limit point of A. Let $x_{n_1} \in S_{r_1}(u) \cap (A - \{u\})$. Then $d(u, x_{n_1}) < r_1$ and $u \neq x_{n_1}$. Now, let $0 < r_2 < \min\{\frac{1}{2}, d(u, x_{n_1})\}$.

Then $\exists\, x_{n_2} \in S_{r_2}(u) \cap (A - \{u\})$ such that

$d(u, x_{n_2}) < r_2 < \frac{1}{2}$ and $x_{n_2} \neq u$. Proceeding in this way, at the k-th step, let $0 < r_k < \min\{\frac{1}{k}, d(u, x_{n_{k-1}})\}$. Then $\exists\, x_{n_k} \in S_{r_k}(u) \cap (A - \{u\})$ with $d(x_{n_k}, u) < r_k < \frac{1}{k} \to 0$ as $k \to \infty$.

Thus $\{x_{n_k}\}$ converges to u and consequently the space (X,d) is sequentially compact. Hence (ii) $\Rightarrow$ (i).

Theorem: If (X,d) is compact then it has B-W property.

proof: Let (X,d) be compact. A be an infinite subset of X. If possible let, no point of X be a limiting point of A. Then for any $x \in X$, $\exists$ an $r>0$ such that $S_r(x) \cap A$ is either empty or contains x alone. clearly the family $\{S_r(x) : x \in X\}$ forms an open cover for X. So, by compactness of (X,d), $\exists$ a finite subcover, say $\{S_{r_1}(x_1), S_{r_2}(x_2), \ldots, S_{r_n}(x_n)\}$ for X. Then $X \subseteq S_{r_1}(x_1) \cup S_{r_2}(x_2) \cup \ldots \cup S_{r_n}(x_n)$ and so

$A \subseteq S_{r_1}(x_1) \cup S_{r_2}(x_2) \cup \ldots \cup S_{r_n}(x_n)$. But $A \cap S_{r_n}(x_n)$ is either empty or $\{x_n\}$. So, we see that A contains at most n elements which contradicts our assumption that A is an infinite subset of X. Thus every infinite subset of X has a limit point in X and Hence (X,d) has B-W property.

Consequently

Theorem: Every Compact metric space is complete.

proof: Let (X,d) be a compact metric space and $\{x_n\}$ be a cauchy sequence in (X,d). Since (X,d) is compact, it is B-W compact and hence is sequentially compact. So, $\{x_n\}$ has a convergent subsequence, say $\{x_{n_k}\}$. Let $\lim_{k\to\infty} x_{n_k} = u \in X$. Let $\epsilon>0$ be preassigned. Since $\{x_n\}$ is a cauchy sequence $\exists$ a positive integer n_1, depending on ϵ such that

$$d(x_m, x_n) < \epsilon/2, \quad \forall\, m,n \geq n_1.$$

Again since $\lim_{k\to\infty} x_{n_k} = u$, corresponding to some $\epsilon>0$, $\exists$ a positive integer n_2 such that

$$d(x_{n_k}, u) < \epsilon/2 \quad \forall\, k \geq n_2.$$

Let $n_0 = \max\{n_1, n_2\}$. Then

$$d(x_k, u) \leq d(x_k, x_{n_k}) + d(x_{n_k}, u)$$
$$< \epsilon/2 + \epsilon/2 \quad \forall\, k \geq n_0 \; (\because n_k \geq k \text{ always})$$

Thus $d(x_k, u) < \epsilon \quad \forall\, k \geq n_0$ and so $\{x_n\}$ is convergent in (X,d). Hence (X,d) is complete.

However the converse of this theorem is not true

For example the real number space w.r.to usual metric is complete. But it is not compact. Because the open cover $\{(n-2, n+2) : n = 0, \pm 1, \pm 2, \ldots\}$ of $\mathbb{R}$ does not admit any finite subcover for $\mathbb{R}$.

Theorem: Every Compact subset in a metric space is closed.

proof: Let G be a compact subset of a metric space (X, d) and u be a limit point of G without being a point of G. Since u is a limit point of G, $\exists$ a sequence $\{u_n\}$ in G such that $\lim_{n\to\infty} u_n = u \notin G$. Since G is compact, it is sequentially compact and so $\{u_n\}$ has a convergent subsequence say $\{u_{n_k}\}$. Then $\lim_{k\to\infty} u_{n_k} = u \in G$, a contradiction. Thus G has no limit point outside it. Hence G is closed.

Theorem: Every Compact Subset of a metric space is bounded.

proof: Let G be a Compact subset of a metric space (X, d). If possible let G be not bounded. Let $x_0 \in X$. Then the open ball $S_1(x_0)$ fails to Contain G. So, $\exists$ a point $x_1 \in G$ such that $d(x_0, x_1) \geq 1$. By similar arguments, we have $x_2 \in G$ such that $d(x_0, x_2) \geq 1 + d(x_0, x_1)$. proceeding in this way at the n-th stage, we find $x_n \in G$ such that $d(x_0, x_n) \geq 1 + d(x_0, x_1) + d(x_0, x_2) + \cdots + d(x_0, x_{n-1})$.

So, for $n > m$ $\quad d(x_0, x_n) \geq 1 + d(x_0, x_m)$ ---- by (1)

Now, by triangle inequality,

$$d(x_0, x_n) \leq d(x_0, x_m) + d(x_m, x_n), \text{ i.e.,}$$

$$d(x_m, x_n) \geq d(x_0, x_n) - d(x_0, x_m) \geq 1 \quad \text{by (1)}$$

This shows that the sequence $\{x_n\}$ in G does not admit any convergent subsequence and so G is not sequentially compact. Therefore G is not compact, a contradiction. So G is bounded and the proof of the theorem is complete.

Theorem: If $f:(X,d)\to(Y,\rho)$ is continuous and (X,d) is compact then $f(X)$ is compact in (Y,ρ).

proof: Let $\{G_\alpha : \alpha\in\Delta$, an index set$\}$ be an open cover for $f(X)$ in (Y,ρ). Since each G_α is open relative to $f(X)$, for each G_α $\exists$ an open set H_α in (Y,ρ) such that

$G_\alpha = H_\alpha \cap f(X)$

Since $f:(X,d)\to(Y,\rho)$ is continuous, $f^{-1}(H_\alpha)$ is open in (X,d) for each $\alpha\in\Delta$. Clearly $X\subseteq \bigcup_{\alpha\in\Delta} f^{-1}(H_\alpha)$ and so $\{f^{-1}(H_\alpha):\alpha\in\Delta\}$ forms an open cover for X. The so, by compactness of (X,d) $\exists$ a finite subcover say $\{f^{-1}(H_1), f^{-1}(H_2), \ldots, f^{-1}(H_n)\}$ to cover X.

Hence $X \subseteq \bigcup_{i=1}^{n} f^{-1}(H_i)$

i.e., $X \subseteq f^{-1}\left(\bigcup_{i=1}^{n} H_i\right)$

So, $f(X) \subseteq \bigcup_{i=1}^{n} H_i$

i.e., $f(X) \subseteq \bigcup_{i=1}^{n}(H_i\cap f(X))$, i.e.,

$f(X) \subseteq \bigcup_{i=1}^{n} G_i$

Therefore $f(X)$ is compact in (Y,ρ).

This proves the theorem.

Corollary: If $f:[a,b]\to R$ is continuous then it is bounded and attains its bounds.

proof: Since $[a,b]$ closed and bounded, therefore by Heine-Borel theorem $[a,b]$ is compact. Therefore the continuous image ~~set~~ $f([a,b])$ is compact in R, the real number space with respect to usual metric. Since $f([a,b])$ is compact, it is closed and bounded. Therefore f is bounded on $[a,b]$. Let m and M be the greatest lower and least upper bounds of $f([a,b])$. If m and M are limiting points of $f([a,b])$, unless they are points of $f([a,b])$. Since $f([a,b])$ is a closed set, m, M belong to $f([a,b])$. So, $\exists$ $u,v\in[a,b]$ such that $f(u)=m$ and $f(v)=M$.

This completes the proof of the corollary.

Definition: A subset B of a metric space (X,d) is said to be an ϵ-net, $\epsilon>0$ is given if

(i) B is a finite set say $B=\{b_1,b_2,\ldots,b_k\}$ and

(ii) $X \subseteq \bigcup_{i=1}^{k} S_\epsilon(b_i)$

Definition: A metric space (X,d) is said to be totally bounded if there exists an ϵ-net for every given $\epsilon>0$.

Theorem: Every totally bounded metric space is bounded but not conversely.

proof: Let (X,d) be a totally bounded metric space and $\epsilon>0$ be given. Then there exists an ϵ-net B say $\{b_1,b_2,\ldots,b_k\}$ such that $X\subseteq\bigcup_{i=1}^{k}S_\epsilon(b_i)$. Let $x,y\in X$. Then $x\in S_\epsilon(b_i)$ and $y\in S_\epsilon(b_j)$ for $1\leq i,j\leq k$

Now,
$$\begin{aligned} d(x,y) &\leq d(x,b_i)+d(b_i,b_j)+d(b_j,y)\\ &< \epsilon+d(b_i,b_j)+\epsilon\\ &= 2\epsilon+d(b_i,b_j)\\ &\leq 2\epsilon+\text{diam}(B) \end{aligned}$$

Since the right side is independent on the choice of x,y in X, we have
$$\sup_{x,y\in X} d(x,y)\leq 2\epsilon+\text{diam}(B)<\infty, \text{ i.e., } \text{diam}(X)<\infty$$
Hence (X,d) is bounded.

However the converse of this theorem is not true. For this let us consider the collection E of all unit vectors e_j, $j=1,2,3,\ldots$ in the space l_2,

where $e_j=(0,0,\ldots,\underbrace{1}_{j\text{th place}},0,0,\ldots)$

The metric on l_2 is given by
$$d(x,y)=\left\{\sum_{i=1}^{\infty}(x_i-y_i)^2\right\}^{1/2} \text{ where}$$
$$x=(x_1,x_2,\ldots,x_n,\ldots)\in l_2 \text{ and}$$
$$y=(y_1,y_2,\ldots,y_n,\ldots)\in l_2$$

Then E becomes a subspace of l_2 with
$$d(e_i,e_j)=\begin{cases}0 & \text{if } i=j\\ \sqrt{2} & \text{if } i\neq j\end{cases}$$

clearly, E is bounded metric space. Let $\epsilon>0$ be such that $0<\epsilon<\frac{\sqrt{2}}{3}$. Then $S_\epsilon(e_i)\cap S_\epsilon(e_j)=\phi$ for $i\neq j$. So for such ϵ there exists no ϵ net. Hence E is not totally bounded.

Definition: Given an open cover $\{G_\alpha : \alpha \in \Delta\}$ of a metric space (X,d), a positive real number r is said to be a Lebesgue number if every subset of X with diameter $< r$ is contained in a member, say G_α of the family $\{G_\alpha : \alpha \in \Delta\}$ entirely.

Theorem: In a sequentially compact metric space every open cover has a Lebesgue number.

proof: Let (X,d) be a sequentially compact metric space and $\{G_\alpha : \alpha \in \Delta\}$ be an open cover for X. We call a subset H of (X,d) as a big set if H is not contained in one member of the family $\{G_\alpha : \alpha \in \Delta\}$. If there exists no big set, then any positive real number can be taken as a Lebesgue number. So, suppose there are big sets in (X,d). Let

$$\lambda = \inf\{\delta(H) : H \text{ is a big set}\}.$$

Clearly, $0 \le \lambda < \infty$. If $\lambda > 0$ then any positive real number less than λ can be taken as a Lebesgue number. finally let $\lambda = 0$. Then from the property of infimum for every positive integer n $\exists$ a big set H_n such that $\delta(H_n) < \frac{1}{n}$. Let $x_n \in H_n$, $n = 1, 2, \cdots$

Now by sequentially compactness of (X,d), the sequence $\{x_n\}$ has a convergent subsequence say $\{x_{n_k}\}$. Let $\lim\limits_{k\to\infty} x_{n_k} = u \in X$. Then $u \in G_\alpha$ for some $\alpha \in \Delta$.

Since G_α is open $\exists$ a real number $r_0 > 0$ such that $S_{r_0}(u) \subseteq G_\alpha$. Again since $\lim\limits_{k\to\infty} x_{n_k} = u$, corresponding to $\frac{r_0}{2} > 0$ $\exists$ a positive integer n_0 such that

$$d(x_{n_k}, u) < \frac{r_0}{2} \quad \forall\, k \geq n_0.$$

Let $k' > n_0$ be such that $\frac{1}{n_{k'}} < \frac{r_0}{2}$.

Let $x \in H_{n_{k'}}$. Then $x_{n_{k'}} \in H_{n_{k'}}$ and so $d(x, x_{n_{k'}}) \le \delta(H_{n_{k'}}) < \frac{1}{n_{k'}} < \frac{r_0}{2}$. Now, $d(x,u) \le d(x, x_{n_{k'}}) + d(x_{n_{k'}}, u) < \frac{r_0}{2} + \frac{r_0}{2} = r_0$

So, $x \in S_{r_0}(u)$. Since x is an arbitrary point of $H_{n_{k'}}$; $H_{n_{k'}} \subseteq S_{r_0}(u) \subseteq G_\alpha$, a contradiction to our assumption that $H_{n_{k'}}$ is a big set.

The proof is now complete.

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Lemma: If (X,d) is sequentially compact then it is totally bounded.

proof: Let $\epsilon>0$ be preassigned. Let $a_1 \in X$. If $X \subseteq S_\epsilon(a_1)$, then $\{a_1\}$ forms an ϵ net. otherwise, let $a_2 \in X - S_\epsilon(a_1)$. Then $d(a_2,a_1) \geq \epsilon$. If $X \subseteq S_\epsilon(a_1) \cup S_\epsilon(a_2)$, then $\{a_1,a_2\}$ forms an ϵ net, otherwise, let $a_3 \in X - \{S_\epsilon(a_1) \cup S_\epsilon(a_2)\}$. Then $d(a_3,a_1) \geq \epsilon$ and $d(a_3,a_2) \geq \epsilon$. If $X \subseteq S_\epsilon(a_1) \cup S_\epsilon(a_2) \cup S_\epsilon(a_3)$, then $\{a_1,a_2,a_3\}$ forms an ϵ net. otherwise we continue this process. Then at the n-th stage we can produce an ϵ net or we produce a sequence $\{a_n\}$ such that $d(a_n,a_i) \geq \epsilon$ for $i=1,2,3,\ldots,n$. clearly, such a sequence does not have any convergent subsequence, which is a contradiction to the assumption that (X,d) is sequentially compact. This proves the lemma.

problem: Let (R,d) be the real number space w.r.to ~~usual~~ ~~metric~~. ~~Find~~ the ~~Lebegue~~ number of the open cover discrete metric. Then for any open cover every positive real number ~~one~~ r satisfying $0<r<1$ is a Lebesque number

Theorem: If (X,d) is sequentially Compact then it is Compact

proof: Let $\{G_\alpha : \alpha \in \Delta\}$ be an open cover of (X,d) which is sequentially compact. But we know that every open Cover of a sequentially compact metric space has a Lebesque number. Hence $\{G_\alpha : \alpha \in \Delta\}$ has a Lebesque number, say $r>0$. Let $\epsilon>0$ be such that $0<\epsilon<\frac{r}{3}$. By the above lemma (X,d) is totally bounded and so for this $\epsilon>0$, $\exists$ an ϵ-net, say $A=\{a_1,a_2,\ldots,a_k\}$. Then $X \subseteq \bigcup_{i=1}^{k} S_\epsilon(a_i)$. clearly every open ball $S_\epsilon(a_i)$ $(i=1,2,\ldots,k)$ has diameter $= 2\epsilon < \frac{2r}{3} < r$ and so by the definition of Lebesque number for each open ball $S_\epsilon(a_i)$ $\exists$ a member G_i from $\{G_\alpha : \alpha \in \Delta\}$ such that $S_\epsilon(a_i) \subseteq G_i$, $i=1,2,\ldots k$.

Therefore $X \subseteq G_1 \cup G_2 \cup \cdots \cup G_k$ and consequently (X,d) is compact.

Theorem: In a metric space (X,d)
(i) finite union of compact sets is a compact set and
(ii) arbitrary intersection of compact sets is a compact set.

proof:
(i) Let $A_1, A_2, \ldots, A_n$ be a finite collection of compact sets in a metric space (X,d) and let $\mathcal{U}$ be an open cover for $\bigcup_{i=1}^{n} A_i$. Then $\mathcal{U}$ is also an open cover for A_i, $i=1,2,\ldots,n$. Since each A_i is compact, there exists a finite subcover $\mathcal{U}_i$ to cover A_i $(i=1,2,\ldots,n)$. Then $\bigcup_{i=1}^{n} \mathcal{U}_i$ is a finite subcover of $\mathcal{U}$ to cover $\bigcup_{i=1}^{n} A_i$. Hence $\bigcup_{i=1}^{n} A_i$ is a compact set in (X,d).

(ii) Let $\{A_\alpha : \alpha \in \Delta\}$ be a family of compact sets in a metric space (X,d) and let $A = \bigcap_{\alpha \in \Delta} A_\alpha$. Since A_α is compact, each A_α is closed in (X,d). Again since arbitrary intersection of closed sets in a metric space is a closed set, A is a closed set in A_α, $\alpha \in \Delta$. Since every closed subset of a compact space is compact, A is compact.

Theorem: (Heine-Borel) Every closed bounded subset in the real line $\mathbb{R}$ is compact.

proof: Let A be a closed and bounded subset of the bounded line $\mathbb{R}$. Since A is bounded, there exists an interval $[a,b]$ such that $A \subseteq [a,b]$. Since every closed subset of a compact space is compact, it sufficies to show that $[a,b]$ is compact

Let $\mathcal{U}$ be an open cover of $[a,b]$ and let $P = \{x \in [a,b] : \mathcal{U}$ has a finite sub cover for $[a,x]\}$. clearly, $a \in P$ and b is an upper bound of P. Thus P has least upper bound, say c. Then $a \leq c \leq b$ and so $c \in [a,b]$. Since $\mathcal{U}$ is an open cover of $[a,b]$, $\exists$ a member U of $\mathcal{U}$ such that $c \in U$. As U is open, $\exists$ an $\epsilon > 0$ such that $(c-\epsilon, c+\epsilon) \subseteq U$. Then by the property of least upper bound, $\exists$ a member x in P such that

$$c-\epsilon < x \leq c.$$

Since $x \in P$, $\exists$ a finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$ to cover $[a,x]$. Then $\mathcal{U}_0 \cup \{U\}$ is also a finite subcollection of $\mathcal{U}$ to cover $[a, c+\epsilon)$. Thus all points of $[a, c+\epsilon) \cap [a,b]$ are in P. But no point of $[a, c+\epsilon]$ which are greater than c are in P. Because c is the least upper bound of P. So, $c \in P$ and $c = b$ ($\because c \leq b$).

Thus $[a,b]$ can be coverd by a finite sub-collection of $\mathcal{U}$. Hence $[a,b]$ is compact and consequently every closed and bounded subset of the real line $\mathbb{R}$ is compact.

NOTE: The word closed and bounded cannot be dropped from Heine-Borel theorem.

The above result also holds in $\mathbb{R}^n$.